Convergence of the Ishikawa Iterative Sequence to Fixed Points of Lipschitz Pseudocontrative Maps in Hilbert Spaces

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Abstract
We study the weak and strong convergence of the Ishikawa iterative sequence to a fixed point of a Lipschitz pseudocontrative mapping , T, in a Hilbert space. We do not require any compactness type assumptions either on T or its domain, for the strong convergence results. Neither do we require that the interior of the fixed points set of T be nonempty, which is a condition used in [25]. Furthermore, we do not need to compute for closed convex subsets, C_n, of the Hilbert space.

MSC: 47H10
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Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to C$ is called:
(i) Lipschitzian if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||,$$

for all $x, y \in C$. If $L < 1$, then T is called a contraction. If $L = 1$, then T is called nonexpansive.
(ii) Pseudocontractive if

$$||x - y|| \leq ||(1 + s)(x - y) - s(Tx - Ty)||$$

for all $x, y \in C$ and $s > 0$, or equivalently

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 \tag{1.1}$$

for all $x, y \in C$
Let $A : D(A) \subseteq E \to E$ be a map. Then A is called accretive if

$$||x - y|| \leq ||x - y + s(Ax - Ay)||. \tag{1.2}$$

for all $x, y \in D(A)$ and $s > 0$. We immediately observe from (1.1) and (1.2) that A is accretive if and only if $T := I - A$ is pseudocontractive, where I denotes the identity operator. Thus, the zeros of accretive operators corresponds to the fixed points of pseudocontrative maps.

Preliminaries

The notion of accretive operators was independently in 1967 by Browder [2] and Kato [1]. An early fundamental result due to Browder, states that the initial value problem

$$\frac{du}{dt} + Au = 0, u(0) = u_0$$

is solvable if A is locally Lipschitzian and accretive on E. Therefore, the importance of research into the fixed point theory of pseudocontractive maps cannot be over-emphasized (given its firm connection with accretive maps). Many authors (see e.g [3], [5], [10], [11], [14], [17]) have contributed considerably to this end.
It is well known that if $T : E \to E$ is a contraction map, then the Picard's iterative sequence, starting from an arbitrary $x_0 \in E$, given by

$$x_{n+1} = Tx_n \tag{1.3}$$

for all $n \geq 0$, converges to the unique fixed point of T. If however, T is a nonexpansive map, then (1.3) is not guaranteed to converge to a fixed point of T, even on a compact subset of E. Observe that if C is the unit disc in $\Re^2$ (which is compact) and T is its rotation about the origin, then T is easily shown to be nonexpansive and has 0 as its unique fixed point. Starting from an x_0 on the circumference, (1.3) does

not converge to the fixed point of T.
Krasnoselskii [18] showed that instead of (1.3), if we consider the averaging sequence

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n)$$

for all $n \geq 0$, then starting from an arbitrary x_0 on the unit disc, we achieve convergence to the fixed point of T. A further generalization due to Schaefer [19] for the fixed points of nonexpansive mappings is

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n$$

for all $n \geq 0,\ \lambda \in (0,1)$.
The most general iteration sequence for nonexpansive mappings which has been studied by many authors is due to W.R. Mann [16] and is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \tag{1.4}$$

for all $n \geq 0,\ \alpha \in (0,1)$, satisfying certain conditions. This iteration sequence, however, does not generally converge to a fixed point of T (when it exists), without additional conditions imposed either on T, the domain of T or the range of T. Without any of these conditions, the best we can get is that $\{x_n\}$ is an approximate fixed point sequence, i.e $\lim ||x_n - Tx_n|| = 0$. To get weak convergence, we need the additional condition that T be demiclosed at zero, together with the fact that E be an Opial space, while to get strong convergence, we need some compactness type assumptions on T, domain of T or range of T.
The natural question that arises is the following:**Can The Mann iterative sequence converge to the fixed points of the more general class of pseudocontractive maps?** To this end, we quickly submit that all attempts to use the Mann iteration sequence for Lipschitz pseudocontractive maps have proven abortive. In [20], Chidume and Mutangadura gave an example of a Lipschitz pseudocontractive self map of a compact convex subset of a Hilbert space with a unique fixed point, for which the Mann iterative sequence fails to converge.
The next natural question is the following: **What iterative sequence can we employ for the convergence to fixed points of Lipschitz pseudocontractive maps?** In [17], Ishikawa introduced an iteration sequence, which in some sense is more general than the Mann iterative sequence, which he used for the convergence to fixed points of Lipschitz pseudocontractive maps. More precisely, he proved the following:
Theorem 1 [17] If C is a compact convex subset of a Hilbert space H and $T : C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n;\quad y_n = (1-\beta_n)x_n + \beta_n Tx_n,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i) 0 \leq \alpha_n \leq \beta_n < 1; (ii) \lim \beta_n = 0; (iii) \sum \alpha_n \beta_n = \infty.$$

The Ishikawa iterative sequence actually leads to an approximate fixed point sequence for Lipschitz pseudocontractive maps, such that the imposition of compactness type assumptions on T or domain of T or range of T, yields convergence to fixed points of T.
In order to obtain strong convergence to fixed points of pseudocontractive maps without the compactness type assumptions, many authors (see e.g [21], [22]) have defined what they call hybrid Mann and Ishikawa algorithms. However, these hybrid schemes are hinged on some special subsets, C_n and Q_n of the Banach space, whose computations are non-trivial.
More recently, Zegeye *et al* [25] proved the following results:
Theorem 2 [25]: Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to C, i = 1, 2, ..., N$ be a finite family of Lipschitz pseudocontractive mappings with Lipschitzian constants L_i, for $i = 1, 2, ..., N$ respectively. Assume that the interior of $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n;\quad y_n = (1-\beta_n)x_n + \beta_n Tx_n,$$

where $T_n := T_{n(mod)N}$ and $\{\alpha_n\},\ \{\beta_n\}\ \in (0,1)$ satisfying the following conditions. $(i)\alpha_n \leq \beta_n\ \forall n \geq 0; (ii) \lim inf\ \alpha_n = \alpha > 0; (iii) sup_{n\geq 0}\beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$ for $L := max\{L_i : i = 1, 2, ..., N\}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, ..., T_N\}$.
Although the results of Zegeye *et al* is plausible, pseudocontraction maps abound whose fixed point sets are finite and as such have empty interiors.
So, the question that still remains to be answered is:

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Is it possible to obtain strong convergence of the Ishikawa iteration sequence (not hybrid) to fixed points of Lipschitz pseudocontraction maps, without the compactness type assumptions on either T or its domain and without the assumption that the interior of the fixed points set be nonempty?

We now state some results in the Literature which will help us answer the above question to a reasonable extent. The first one and its proof is given by Zhou in [21], as **Tool 2.**

Tool 2 Zhou [21](Demiclosedness): Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ be a demicontinuous pseudocontractive self mapping from C into itself. Then $F(T)$ is a closed convex subset of C and $I - T$ is demiclosed at zero.

Lemma 1 [26]: Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1 + \delta_n)a_n + b_n, n \geq 1$. If $\sum \delta_n < \infty$ and $\sum b_n < \infty$, then $lim\ a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $lim\ a_n = 0$.

Theorem 3 (Maruster [23]): Let $T : C \to C$ be a nonlinear mapping with $F(T) \neq \emptyset$, where C is a closed convex subset of a real Hilbert space H. Suppose the following conditions are satisfied:
(i) $I - T$ is demiclosed at 0
(ii) T is demicontractive with constant k, or equivalently T satisfies condition A with $\lambda = \frac{1-k}{2}$
(iii) $0 < a \leq \alpha_n \leq b < 2\lambda = 1 - k$
Then the Mann iteration sequence converges weakly to a fixed point of $F(T)$, for any starting x_0.

Theorem 4 (Maruster [23]): Suppose T satisfies the conditions of theorem 3. If in addition there exists $0 \neq h \in H$ such that

$$\langle x - Tx, h\rangle \leq 0 \tag{1.5}$$

for all $x \in D(T)$, then starting from a suitable x_0, the Mann iteration sequence (1.4) converges strongly to an element of $F(T)$.

In [24], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq ||x||^2$, then it is sufficient to find a non-zero solution of the variational inequality (1.5). This motivates our choice of monotonicity type condition.

It is our purpose in this article to prove weak and strong convergence of the Ishikawa iterative sequence to a fixed point of a Lipschitz pseudocontractive map in a nonempty closed convex subset of a Hilbert space. We do not need any compactness type assumption on T or its domain. Neither do we require that the interior of the fixed points set of T be nonempty.

Before we state and prove our main results, we give a definition which will be useful in the sequel.

Definition 1: Let H be a real Hilbert space with inner product $\langle ., .\rangle$ and norm $||.||$ and let C be a nonempty closed convex subset of H. The orthogonal projection $P_C x$ of x onto C is defined by $P_C x = arg\ min_{y \in C}||x - y||$, and has the following properties:
(i) $\langle x - P_C x, z - P_C x\rangle \leq 0,$ for all $z \in C$
(ii) $||P_C x - P_C y||^2 \leq \langle P_C x - P_C y, x - y\rangle,$ for all $x, y \in H$

Main Results

Theorem 5: If C is a closed convex subset of a real Hilbert space H and $T : C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges weakly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n; \quad y_n = (1 - \beta_n)x_n + \beta_n Tx_n,$$

where $\{\alpha_n\},\ \{\beta_n\}\ \in (0, 1)$ satisfy the following conditions. (i) $\alpha_n \leq \beta_n$ $\forall n \geq 0$;
(ii) $inf\ \alpha_n = \alpha > 0; (iii) sup_{n\geq 0}\ \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Proof

As in the proof of our theorem 1 in [17], we have

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n\beta_n(1 - 2\beta_n - L^2\beta_n^2)||x_n - Tx_n|| \tag{1.6}$$

This, together with the conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$ yields $lim||x_n - Tx_n|| = 0$.
From (1.6) and lemma 1, we have that $lim\{||x_n - p||\}$ exists. It follows that $\{||x_n - p||\}$ is bounded. Therefore $\{x_n\}$ is norm bounded. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakyly to $x^* \in C$. These, together with **Tool 2** implies $x^* \in F(T)$. Since H is an Opial space, a well known standard argument yields that $\{x_n\}$ converges weakly to x^*.

Theorem 6: If C is a closed convex subset of a Hilbert space H and $T : C \to C$ is a Lipschitz pseudocontractive mapping and x_0 is any point of C, then the sequence $\{x_n\}_{n\geq 0}$ converges weakly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n; \quad y_n = (1 - \beta_n)x_n + \beta_n Tx_n,$$

where $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$ satisfy the following conditions. $(i)\alpha_n \leq \beta_n \; \forall n \geq 0$; $(ii) \sum \alpha_n\beta_n = \infty (iii) sup_{n\geq 0}\beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$.

Proof

As in the proof of our theorem 1 in [17], we have

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n\beta_n(1 - 2\beta_n - L^2\beta_n^2)||x_n - Tx_n|| \qquad (1.7).$$

This, together with the conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$ yield $liminf||x_n - Tx_n|| = 0$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $lim||x_{n_k} - Tx_{n_k}|| = 0$.

From (1.7) and lemma 1, we have that $lim\{||x_n - p||\}$ exists. It follows that $\{||x_n - p||\}$ is bounded. Therefore, $\{x_n\}$ is norm bounded. Since $\{x_n\}$ is norm bounded, so is $\{x_{n_k}\}$ and as such, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakyly to $x^* \in C$. These, together with **Tool 2** implies $x^* \in F(T)$. Since H is an Opial space, a well known standard argument yields that $\{x_n\}$ converges weakly to x^*.

Theorem 7: Let C be a nonempty closed convex subset of a real Hilbert space, H. Let $T : C \to C$ be a Lipschitz pseudocontractive with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Suppose T satisfies the conditions of either theorem 5 or 6 and the monotonicity condition $\langle Tx - x, p\rangle \geq 0, \forall x \in C, p \in F(T)$. Then starting from a suitable $x_0 \in C$, the Ishikawa iteration sequence converges strongly to an element of $F(T)$.

Proof Let $p \in F(T)$. Choose $x_0 \in C$, such that $\langle x_0, p\rangle \geq \langle p, p\rangle$. Then, there exists $\epsilon_0 > 0$ such that $\langle x_0 - p, p\rangle \geq \epsilon_0||x_0 - p||^2$. Assume $\langle x_n - p, p\rangle \geq \epsilon_0||x_n - p||^2$. Then using the monotonicity condition in our theorem and (1.7), we have

$$\begin{aligned}
\langle x_{n+1} - p, p\rangle &= \langle[(1-\alpha_n)x_n + \alpha_nTy_n] - p, p\rangle \\
&= \langle(1-\alpha_n)[x_n - p] + \alpha_n[Ty_n - p], p\rangle \\
&= (1-\alpha_n)\langle x_n - p, p\rangle + \alpha_n\langle Ty_n - p, p\rangle \\
&= (1-\alpha_n)\langle x_n - p, p\rangle + \alpha_n[\langle y_n - p, p\rangle + \langle Ty_n - y_n, p\rangle] \\
&= (1-\alpha_n)\langle x_n - p, p\rangle + \alpha_n\langle(1-\beta_n)[x_n - p] + \beta_n[Tx_n - p], p\rangle \\
&\quad +\alpha_n\langle Ty_n - y_n, p\rangle \\
&= (1-\alpha_n)\langle x_n - p, p\rangle + \alpha_n(1-\beta_n)\langle x_n - p, p\rangle \\
&\quad +\alpha_n\beta_n\langle Tx_n - p, p\rangle + \alpha_n\langle Ty_n - y_n, p\rangle \\
&= (1-\alpha_n)\langle x_n - p, p\rangle + +\alpha_n(1-\beta_n)\langle x_n - p, p\rangle \\
&\quad +\alpha_n\beta_n[\langle x_n - p, p\rangle + \langle Tx_n - x_n, p\rangle] + \alpha_n\langle Ty_n - y_n, p\rangle \\
&= \langle x_n - p, p\rangle + \alpha_n\beta_n\langle Tx_n - x_n, p\rangle] + \alpha_n\langle Ty_n - y_n, p\rangle \\
&\geq \langle x_n - p, p\rangle \\
&\geq \epsilon_0||x_n - p||^2 \\
&\geq \epsilon_0||x_{n+1} - p||^2
\end{aligned}$$

So that, since $x_n \rightharpoonup p$ from theorems 5 and 6 then $x_n \to p$.

Example 1: Let $H = R$ (reals) and $C = [1,2]$ be a nonempty closed convex subset of H. Define $T : C \to C$ by

$$Tx = \begin{cases} 1, & if \;\; 1 \leq x \leq \frac{3}{2} \\ \frac{3}{2}, & if \;\; \frac{3}{2} < x \leq 2 \end{cases}$$

Then T is a pseudocontractive mapping with a non-empty fixed points set and satisfies $\langle x - Tx, p\rangle \geq 0$,for all $x \in [1,2]$. To see this, observe that $F(T) = \{1\}$ and $\langle x - Tx, p\rangle \geq 0$, for all $x \in [1,2]$. Furthermore,
(i) For $1 \leq x \leq \frac{3}{2}$, we have

$$\begin{aligned}
||Tx - p||^2 = |1 - 1|^2 &= 0 \\
&\leq |x - 1|^2 + |x - Tx|^2 \\
&= ||x - p||^2 + ||x - Tx||^2
\end{aligned}$$

(ii) For $\frac{3}{2} < x \leq 2$, we have

$$\begin{aligned} ||Tx-p||^2 = |\frac{3}{2}-1|^2 &= \frac{1}{4} \\ &< \frac{1}{4} + |x-Tx|^2 \\ &\leq |x-1|^2 + |x-Tx|^2 \\ &= ||x-p||^2 + ||x-Tx||^2 \end{aligned}$$

From these two cases, we see that T is a pseudocontractive mapping with a non-empty fixed points set.

Example 2: Let $H = R$ (reals) and $C = [1, 2]$ be a nonempty closed convex subset of H. Define $T : C \to C$ by $Tx = 1$. Then $F(T) = \{1\}$ and it is easily verifiable that T satisfies all the conditions of theorem 7. Therefore, the class of maps for which our results hold is non-void.

Remark 1: In [24], Maruster and Maruster discussed several ways of choosing x_0. One other way of choosing x_0 is the following: For any $\beta > 1$, choose $x_0 = P_C(\beta p)$, where $p \in F(T)$ and $P : H \to C$ is the metric projection from H into C. This follows since it is well known (see Definition 1) that P is firmly nonexpansive (i.e satisfies condition (ii) of Definition 1), so that

$$\begin{aligned} ||x_0-p||^2 &= ||P_C(\beta p) - P_(p)||^2 \\ &\leq \langle P_C(\beta p) - P_C(p), \beta p - p\rangle \\ &= \langle x_0 - p, (\beta-1)p\rangle \\ &= (\beta-1)\langle x_0 - p, p\rangle \end{aligned}$$

This implies $\langle x_0 - p, p\rangle \geq \epsilon_0||x_0 - p||^2$, where $\epsilon_0 = \frac{1}{\beta-1}$

Remark 2: Maps such as the one in our example 2 above have singleton or finite fixed points set and as such do not have non-empty interiors. Hence, the convergence results in [25] do not work for such maps. Therefore, our results among other things, complement the results in [25].

Remark 3 : In [24], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq ||x||^2$, then it is sufficient to find a non-zero solution of the variational inequality (1.5). This motivates our choice of monotonicity type condition in theorem 7, which helps in proving strong convergence results for pseudocontractive maps.

Remark 4: Observe that prior to the work embodied herein, the methods employed in [24] have never been used for the class of pseudocontractive maps.

Conflict of Interest: The author declares that there is no conflict of interests surrounding this research.

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